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Properties of the Directional Derivatives and Gradient Vector

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We know that f' carries important information about the original function f. In one example we saw that $f'(x)$ tells us how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time *x*. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

Recall that if $z = f(x,y)$, then the partial derivatives f_x' and f_y' are defined as

$$f_x'(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \qquad (1)$$

$$f_y'(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

and represent the rates of change of *z* in the *x*- and *y*- directions, that is in the directions of unit vectors ***i*** and ***j***.

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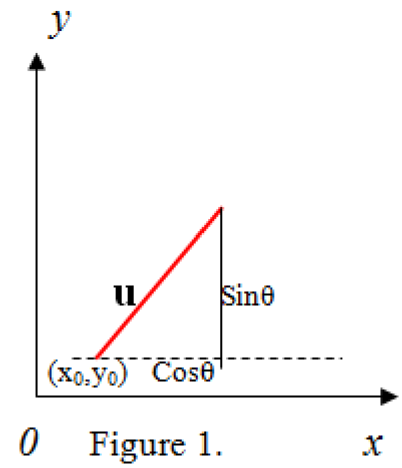

Figure 1.

Suppose that we now wish to find the rate of change of ***z*** at (x_0, y_0) in the direction of an arbitrary unit vector $u = \{a, b\}$ (Figure 1). To do this we consider the surface S with **u** Sinθ equation $z = f(x, y)$ and we let (x0,y0) Cosθ $z_0 = f(x_0, y_0)$. *0* Figure 1. *x* Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction **u** intersects S in a curve C. (Figure 2). The slope of the tangent line T. The slope of the tangent line T to C at P is the rate of change of ***z*** in the direction ***u***. If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q on the *xy* plane, then the vector $P'Q'$ is parallel to **u** and so

$$\overrightarrow{P'Q'} = hu = \{ha, hb\}$$

for some scalar *h*. Therefore, $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$ and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of ***z*** in the direction of **u,** which is called the directional derivative of *f* in the direction of **u.**

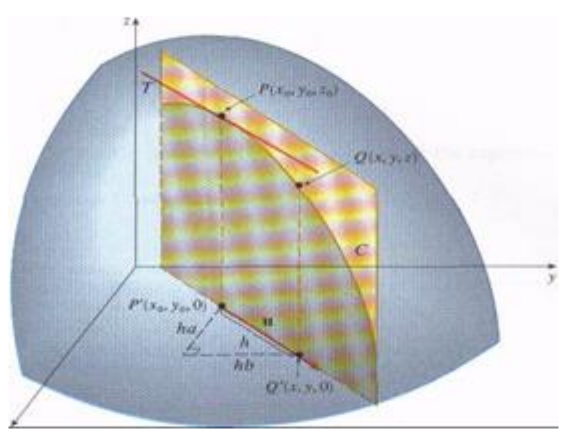

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Figure 2 Definition. The directional derivative of *f* at (x_0, y_0) in the direction of a unit vector $u = \{a, b\}$ is $D_u f(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$ if this limit exists.

By comparing this definition with equations (1) we see that if $u = i = \{1, 0\}$, then $D_i\ f = f_x$ and if $u = j = \{0, 1\}$, then $D_j\ f = f_y$. In other words, the partial derivatives of *f* with respect to *x* and *y* are just special cases of the directional derivative. For computational purposes we generally use the formula given by the following theorem.

If *f* is a differentiable function of *x* and *y*, then *f* has a directional derivative in the direction of any unit vector $u = \{a, b\}$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b \qquad (3)$$

If the unit vector makes an angle θ with the positive *x* - axis, then we can write **u** = {Cos θ, Sin θ} and the formula (3) becomes

$$D_u f(x, y) = f_x(x, y)\mathrm{Cos}\,\theta + f_y(x, y)\mathrm{Sin}\,\theta \qquad (4)$$

Gradient. If *f* is a function of two variables *x* and *y*, then the **gradient** of *f* is the vector function ∇f defined by

$$\nabla f(x, y) = \{\ f_x(x, y), f_y(x, y)\ \} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \qquad (5)$$

Example 1. If $f(x, y) = \mathrm{Sin}\, x + e^{xy}$, then

$\nabla f(x, y) = \{ Cosx + ye^{xy},\ xe^{xy} \}$ and $\nabla f(0,1) = \{ 2,\ 0 \}$.

With this notation for gradient vector, we can rewrite the expression for directional derivative as

$$D_u f(x, y) = \nabla\ f(x, y) \cdot u \quad (6)$$

Example 2. Find the directional derivative of the function $f(x, y) = x^2\ y^3 - 4y$ at the point (2, -1) in the direction of the vector $v = 2i + 5j$.

Solution: We first compute the gradient vector at (2, -1);

$$\nabla\ f(x, y) = 2x\,y^3\,i + \left(3x^2 y^2 - 4\right) j$$

and $\nabla f(2, -1) = -4i + 8j$

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Note that ***v*** is not unit vector, but since $|v| = \sqrt{29}$, the unit vector in the direction of ***v*** is

u = **v**/$|v|$ = $\frac{2}{\sqrt{29}}$ **i** + $\frac{5}{\sqrt{29}}$ **j**

Therefore, by the equation (6), we have

$$D_u f(2,-1) = \nabla f(2,-1) \cdot u = (-4i+8j) \cdot \left(\frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j\right) = \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

For functions of three variables we can define directional derivative in a similar manner. That is the following proposition is true:

The directional derivative of *f* at (x_0, y_0, z_0) in the direction of ***a*** unit vector

$u = \{a, b, c\}$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h} \tag{7}$$

if this limit exists.

If we use vector notation, then we can write both definitions of the directional derivative in the compact form

$$D_u f(x_0) = \lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h} \tag{8}$$

Where $x_0 = (x_0, y_0)$ if n = 2 and $x_0 = (x_0, y_0, z_0)$ if n = 3. This is reasonable because the vector of the line through x_0 in the direction of the vector ***u*** is given by $x = x_0 + hu$ and so $f(x_0 + hu)$ represents the value of *f* at a point on this line.

If $f(x, y, z)$ is differentiable and $u = \{a, b, c\}$, then the following formula is true

$$D_u f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c \quad (9)$$

For a function *f* of three variables, the **gradient vector**, denoted by ∇f or **grad f,** is

$$\nabla f(x,y,z) = \{ f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \} \quad (10)$$

or, for short,

$$\nabla f = \{ f_x,\ f_y,\ f_z \} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \quad (11)$$

Then, just as with functions of two variables, Formula (9) for directional derivative can be rewritten as

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$$

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The directional derivative represents the rate of change of ***z*** in the direction **u.** This is the slope of the tangent line to the curve of intersection of the surface and the vertical plane through the point $P(x_0, y_0, z_0)$ in the direction of **u**.

The upshot is that this problem, finding the speed of something, is exactly the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince us that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the rate at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Literature

1. K.F.Riley, M.P.Hobson, S.J.Bence. Mathematical methods for physics and engineering.– Cambridge, 2006, 1305 p.
2. Ильин В.А., Садовничий В.А., Сендов Б.Х. Математический анализ 2- том. Москва.: Наука 1998.

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