AMERICAN JOURNAL OF SOCIAL AND HUMANITARIAN RESEARCH

ISSN: 2690-9626
Vol. 3, No. 12, 2022

The Axiomatic Method and Model Theory in Mathematics and Logic: An Expository Analysis

Etorobong Godwin Akpan
Department of Philosophy University of Port Harcourt
etorobong.akpan@gmail.com

ABSTRACT: The paper is concerned with the investigation of the concepts of axiomatic method and model theory. The purpose of the paper is an expository one. The study was warranted by the wanton approach in mathematical logic textbooks, where authors punish readers by presupposing a knowledge base that the latter do not possess. The methods adopted for the study are content analysis and the traditional historical approach. The paper has demonstrated the features of the axiomatic method and how such features have evolved over time, as well as their application in formalisms. It has also been shown that model theory, just as proof theory, is a necessary corollary of the axiomatic method. The paper concludes by presenting a sample model of an abstract formalism.

KEYWORD: axiomatic method, mathematical logic, model theory, interpretation, and bijection.

Introduction

Many introductory textbooks of mathematical logic are written with excessive presumptions in mind by their writers. Authors assume that readers will somehow have come into contact with some of the major concepts in the discipline before meeting their texts. Some authors are even so ambitious that they will start a so-called textbook in mathematical logic by presenting a formal language of their own with the intent to use such a language to introduce the discipline. The reason for this tendency is often not clear. Given that mathematical logic is a relatively new discipline, any introductory textbook to it should be down-to-earth and extremely explanatory, without any presumptions, because most of those venturing into the area of study may not have had prior contact with it. Besides, why would a book be called an introductory text when it has no intention to do so?

Here is the big surprise. Most of the prominent practising logicians, mathematicians, and computer scientists have never come across the majority of the concepts used in mathematical logic. So, to apply such concepts in formalistic ways without explanation is a misnomer. Even when mathematical logic textbooks are written with mathematics or computer science majors in mind, some of the concepts are ones that such students have not encountered during their studies. Given that mathematical logic draws its contents from logic and mathematics, it would be wrong to write such textbooks, presuming that there is no further need for explanation if the book is going to serve either a philosophy or mathematics major. Apart from that, all mathematics or philosophy majors around the world do not possess the same level of knowledge of their discipline as is assumed by some of these authors. Hence, it would amount to futility to write a textbook of

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

mathematical logic by presuming that all readers would have known what the author knows prior to writing the book.

Another reason mathematical logic textbooks are inaccessible is because of the quest for sophistication. The search for sophistication would be necessary if the only value of mathematical logic is formalistic elegance. But the discipline has garnered applications in computer science and information technology. Hence, there is a need for in-depth exposition of its concepts at all times of its presentation.

The paper has taken a lead in this direction by setting out to deliver an exposition of two of the main concepts in mathematical logic, namely, the axiomatic method and model theory. The methods adopted for the study are the traditional historical approach and content analysis.

Evolution of the Axiomatic Method

It is now widely accepted that the scientific heritage, which is the foundation of human development, is the product of many civilizations, ranging from Egyptian to Greek, Hindu, and Roman to the current western-dominated world, with each contributing its own fragment. According to their historical writers, Egyptian and Babylonian mathematics were empirical in nature, serving only the needs of their times. The transformation of knowledge into a deductive scientific structure was due to the Greeks, who found leisure in seeking knowledge for its own sake. The first stage of Greek deduction was carried out on geometry by Thales and on the numbers of the Pythagorean School.

The evolving practise gave structure to the Greek mind, such that the deductive practise as a methodology existed with Zeno and featured prominently in Plato’s Academy (Wilder, 1955, p. 4). Thus, Aristotle is quoted as having laid down the structure of the deductive structure as follows:

Every demonstrative science must start from indemonstrable principles; otherwise, the stages of demonstration would be endless. Of these indemonstrable principles some are (a) common to all sciences; others are (b) particular or peculiar to the particular science. In (a), the common principles are the axioms, most commonly illustrated by the axiom that, if equals are subtracted from equals, the remainder are equals. In (b), we have first the genus or subject matter, the existence of which must be assumed (Wilder, 1955, pp. 3–4).

Aristotle also developed his logic, which laid down the principles of demonstration, and he called it syllogism. The *Elements* of Euclid appeared around the time Aristotle was writing, employing all of the principles advocated by the former. Euclidean geometry divided its basic propositions into two main groups, namely, the axioms and the postulates. The axioms fulfilled the requirements of the propositions of the first type as explained by Aristotle, while the postulates satisfied those of the second type. Euclid deduced about 465 propositions from the postulates (Wilder, 1955, p. 4). Such that: "... although all the assertions (are) empirically true, there is no appeal to experience for justification: (Euclid) proceeded only by way of demonstration, basing his proofs only on what has been previously established and obeying the laws of logic alone" (Blanche’, 1965, p. 1).

The approach made the Euclidean system a paradigm of the deductive system. The project made Leibniz come to the conclusion that the Greeks were people who reasoned with justice (Blanche, 1965, p. 2). Thus, in teaching geometry to schoolchildren, Euclid came to be appreciated more as a teacher of logic than of geometrical truths.

The discovery of defects in the Euclidean system, which dates back to the 5th century at the hands of Proclus, resurfaced in the rigours of the Renaissance revolution. However, it received great attention in the late 19th and early 20th centuries from David Hilbert. The revolution was due to the definition of axiomatic rigour by M. Pasch in 1882 (Wilder, 1955, p. 7).

ISSN 2690-9626 (online), Published by “Global Research Network LLC"
under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

The new rigour for the axiomatic method gave rise to the new geometries that are non-Euclidean, in the hands of Bolyai, Lobachevski, and Gauss, in which the fifth axiom of Euclid failed to apply. The resulting new era of the axiomatic method moved emphasis from meaning to logical relations among terms.

Formalism surpassed the era of non-Euclidean geometries in the development of axiomatic methods. Understood as the time of formalised axioms, this era was bent on overcoming the defects of intuition from which ordinary deductive theory suffered. The main feature of the time was the detachment of the formal system from all theoretical concepts. These concepts were replaced by meaningless symbols. Established algorithms replaced the obvious logical rules. The resultant system was a physical calculus possessing finitist characteristics. This period saw the flourishing of the works of Bertrand Russell and Alfred Whitehead, David Hilbert, Paul Bernays, etc., and it is the last stage of the development of the axiomatic methods.

The Structure of the Axiomatic Method

What has been described above could be said to represent three different methodological structures of the axiomatic method. The first has already been stated by Aristotle as consisting of:

1. Axioms
2. Postulates
3. Theorems
4. Rules of logical inference
5. Undefined terms known intuitively (Blanche, 1955, p. 20).

Blanche' describes the second as follows:

1. Explicit enumeration of the primitive proposition for subsequent use in definitions
2. Explicit enumeration of the primitive proposition for subsequent use in demonstrations
3. The relations between the primitive terms shall be purely logical relations, independent of any concrete meaning, which may be given to the term.
4. These relations alone should occur in the demonstrations, and independently of the meaning of the terms so related (this precludes, in particular, relying in any way on the diagram) (Blanche', 1955, p. 22).

The superiority of the present system to the Aristotelian system presupposed in Euclidean geometry lies in its lack of appeal to non-logical intuitions like the intuition of space. However, as previously stated, the presence of logical intuitions allowed the introduction of untested theoretical assumptions into the system. Such assumptions could be about the truth or validity of either mathematics or logic.

In order to overcome these limitations, the formalist evolved an axiomatic method that depends solely on the operations of meaningless symbols by virtue of some algorithms stated at the outset of the construction of the system.

All formal systems of such orientations possess the following properties, according to A. G. Hamilton:

1. An alphabet of symbols
2. A set of finite strings of these symbols called "well-formed formulas." These are to be thought of as the words and sentences in our formal language.
3. A set of well-formed formulas, called axioms.
4. A finite set of 'rules of deduction', i.e., rules which enable one to deduce a set of well-formed formula A, say as a 'direct consequence' of a finite set of well-formed formulas A_1,... A_k, say (1978, pp. 27 - 28).

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

Hamilton did not mention a rule stating how to construct a well-formed formula from the primitive terms, so that *(2.)* above could be arrived at. The moment all these properties are set, a formal system is presupposed. The rule of proof theory would not at first be about interpretation as much as it would be about satisfaction. As a result, using the stated algorithms, it is possible to determine whether a proposition x, y, z, etc. is a system theorem or not. The first concern of formalists is not the logical consistency of their propositions but the formal theoremhood of such propositions in a given system.

Model Theory as a Corollary of the Axiomatic Method

Model theory is a necessary corollary of developments in axiomatics, especially with respect to satisfying formal systems. A model is the realization, interpretation, or satisfaction of a formal theory by a defined domain of objects. So, the first appearance of model theory could be traceable to the investigation of axiomatic systems. It was at the turn of the 20th century that some writers in the foundations of mathematics began to emphasise that any deductive system must be capable of arbitrary interpretation (Mancosu et al., 2001, p. 117), with the only restriction that the primitive sentences are satisfied by the particular interpretation. The goals of the interpretations were the verification of the truth, consistency, and independence of the axioms.

The first tradition, which prominently emphasised model-theoretic interpretation, was that of the algebra of logic. It is to this tradition that we owe what is considered the very first important result in model theory (as we understand it today, i.e., a formal study of the relationship between language and its interpretations). Skolem (1970, p. 107) states the theorem thus: "... Lowenheim proved an interesting and very remarkable theorem on what are called "first-order expressions."... The theorem states that "every first-order expression is either contradictory or already satisfiable in a denumerably infinite domain".

The concepts of satisfaction and domain were prominent in this tradition. Lowenheim and Skolem used the concept of satisfaction rather than interpretation. However, classification of semantic notions was not a major issue in the early stages of logic development. When Ajdukiewicz (1966) wrote "From the Methodology of Deductive Sciences" (p. 9), he was more concerned with the philosophical problems of truth and satisfaction of axiomatic systems than the development of a clear science of semantics.

The gradual systemization of semantics can be traced back to an article published by Bocher in which interpretation was provided by systems of objects with a relation defined on top of it (Bocher, 1904, p. 128).The idea of a "model" is due, in part, to von Neuman, who used the concept to discuss satisfaction for theory. Weyl's publicity and application were responsible for its widespread influence and application. He used the concept of "model" to introduce techniques for proving the independence of axioms and for constructing models of axiom systems. The near universal reception that was accorded the concept and its attendant representations in Weyl gave rise to the formal science of model theory.

The General Concept of Model and Modelling

Model construction is not unique to mathematics. The concept of "model" does not necessarily denote interpretation in ordinary usage. A "model" is sometimes understood as an image or a copy of the real thing. The imagery concept of a model describes standards that may not be perfectly copied by the original. Under this class, we have scale models, paradigms or models as ideals, models as modality, and simulations or working models (Thompson, 1990, p. 67).

A scale model is an abstract descriptive feature of an original that is capable of promoting its isomorphic representation in many realizations. An instance of a scale model could be a locomotive engine or aircraft. The essence of such model construction could be to enhance explanation or present a sketchy idea of what the system means. In such an instance, it is advised that the model should only use very few features of the original (Thompson, 1990, p. 67).

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

The paradigmatic or ideal model theory understands the model as a pattern, an example, a paradigm or an ideal to imitate, a plan to follow, something on which we model ourselves, or something we are making. In this sense, we can refer to a "model husband" (Thompson, 1990, p. 67). A model can sometimes be the real thing, taken as a standard to be emulated by all others. Another example is the new tradition of advertisement modeling, in which good fortune, healthy skin, and a relaxed lifestyle are presented as the consequences of using a particular product.

What differentiates the paradigmatic model from the model as modality is that the former is built on the foundation of an existing standard or kind, whereas the latter is a statement of possible ideal realizations. The concept of modality derives from the concept of possibility as a model. So, this type of model tries to capture what might be but is not.

A simulation or working model is one that is created not just to show a picture of a given fact (isomorphic or not), but also to show how a system works. This model is seen in a situation where a model railroad has objectives for both what the railroad looks like and how it works. The model is made to represent the interactive events in a normal working situation. The simulation model is carried out by means of the computer. Even though the model situation may be different from the realization, there is a close relationship between them if all variables are carefully considered.

Another class of models represents those models, which in this instance are language models. Language is, by its nature, symbolic and abstract. Some others are interpretations or realizations of some kind of abstract structure. There is a general belief that language gains its meaning from referents or the senses of the speech community, even though the notion of the senses of speech is suspect (Resnik, 1980, p. 19). Language's model-theoretic structure is perfectly captured by the nature of scientific theories. Such theories are isomorphic presentations of patterns in physical or mathematical realities. Thus, theories are understood as tools for explaining and showing causal relationships among entities.

An example of the interpretative model is the mathematical model. "Mathematicians use the term "model" to refer to interpretations, in concrete terms, of an abstract set of relations" (Thompson, 1990, p. 77). The relationships are defined in an axiomatic system, which shows the relationship between deducing a set of propositions from a set of axioms. Ancient and modern writers had a different understanding of the epistemological status of the axioms than their contemporary counterparts. The latter understood the axiomatic system as a system of games possessing both rules and moves. The axioms were taken as the rules of the game, whereas the theorems or propositions were taken as moves. Euclidean geometry, which was a paradigm of the axiomatic method until the evolution of non-Euclidean geometry, considered logical principles as the axioms of the system and geometrical statements as its propositions. Contemporary development in the axiomatic method, especially in the hands of the formalist school, has separated the rules from the actual operations of the system. Thus, axioms are themselves propositions from which other propositions called theorems are derived.

The axiomatic system is often considered a meaningless set of propositions in need of interpretation. In most cases, such interpretations are isomorphic, especially for second-order systems. The concepts of points and lines were used in the interpretation of Euclidean geometry (Thompson, 1990, p. 77). The project of presenting interpretations and studying their relationships to the axiomatic system itself forms a complex discipline called model theory.

The Concept of a Model for an Axiomatic System

Apart from the ancient axiomatic method, championed by Aristotle and Euclid and characterised by intuitive understanding of concepts and possessing no meaningless terms, modern and contemporary axiomatic methods have either relative or absolute meaninglessness, thereby standing in need of interpretation. The

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

provision of meaning for the abstract logical structures that characterise these systems is known as interpretation. The meaning given to an axiomatic theory is the understanding of the formal system as a logical frame for one or more concrete theories. This concrete realisation of an axiomatic system is called a "model" (Blanche', 1965, p. 36).

A model is actually a concrete theory that matches the logical structure of the axiom system. A system could have more than one model. Such that:

... the original concrete theory, the one that furnished the data for the logically structured outline by axiomatization, will be one of the models, but not the only one. An axiomatization thus lends itself to different realizations, which can be taken from fields of study very far removed from the initially given domain. Thus, what we are now concerned with is a plurality of interpretations or concrete models of one and the same axiomatization (Blanche', 1990, p. 36).

When models are different from each other in virtue of their interpretation while retaining similar abstract logical structure, they are called isomorphic (Blanche', 1965, p. 36). Isomorphic models could be said to be monomorphic if each of them could be interpreted as a model for the other. That is possible if all models of the system are unique models as the main interpretation of the system and others are only models as long as the basic model serves as a reference point. Once such a model exists, an axiom system is said to be strongly complete.

From the foregoing analysis, a monomorphic model is not actually isomorphic. It is a one-of-a-kind model that models an axiom system in such a way that other systems become isomorphic by possessing its unique elements, thereby only realising the main system by realising its one-of-a-kind model. Jane Bridge (1977) believes that such systems are non-isomorphic (p. 113). She prefers to call it an elementary equivalence (p. 113). Some scholars allow elementary equivalence for more than one class of models. At that point, weak model completeness could be tolerated (Blanche', 1965, p. 37). But where strong model completeness is involved, the model is necessarily singular, thereby modeling other classes on the basis of itself.

Sample Model for an Abstract Formalism

Modeling as such is a bijection that pairs each term in the axiom system with an entity in the model. It can be interpreted as giving meaning to a term that would otherwise be meaningless in the system. We shall now consider some examples of mathematical models. The method of exposition will involve first a formalistic analysis and then a model-theoretic interpretation.

The system that we shall study is to be called System *P*. The objective of this study is to show how the principle of bijection is exposed in axiomatic system modelling (Mancuso et al., 2003, p. 68). The present system is founded on definitions presented in Tarski's *A Decision Method for Elementary Algebra and Geometry* (1951) and Jane Bridge's *Beginning Model Theory* (1977).

The following truth is presupposed in the presentation of the analysis: "A first-order language *L*, even though associated with a given (type of) structure, is technically a syntactic object with no semantic significance" (Bridge, 1977, p. 23). The intention, however, is to be able to express propositions concerning a given structure. The model analysis that follows immediately is based on Bridge's presentation (p. 23). With respect to the model, the variables are to be interpreted as ranging over the domain *Q*. The constant symbols and function symbols are interpreted as the constants and functions with which they are in one-to-one correspondence. The variable structure L(P) and the domain Q have the aforementioned correspondence. So the terms, relative to the assignment of elements of *Q* to the variables, are interpreted as elements of *Q* (i.e., they are viewed in terms of *Q* - realization of *P*). The predicates are interpreted as the corresponding relation between *P* and *Q*, and the logical connectives are allowed to retain their intended meanings as follows: '¬' is

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

'not', '&' 'and', and '∀x' is 'for every element of Q'. A given formula ϕ is interpreted, relative to an interpretation of the free variables, as assertions about Q (p.23).

The informal ideas are presently illustrated by the use of formal definitions. Remember that the language structure to be modelled is L (P*)*. Given any formula $x_1 \ldots x_2$, in *Q*, the interpretation of *x* in *P* is such that the formula variable y_1 is replaced by the domain variable x_1 such that y_1 has the intended meaning; $x \in Q$ *and* $y \in P$. In the same way function formulas of one system could be replaced by those of another. For instance, ϕx could be replaced by another function formula ψy. The principle explained here is called bijection.

From the foregoing analysis, it could be stated that model theory is the provision of interpretation of an axiom system by virtue of another system, such that all the individuals, the constants, the functions, the relations, the formulas, and the connectives of the axiom system would possess a meaning relative to the model. Thus, an adequate model for an axiomatic system is one that completely translates all the properties of, say, a system L (P*)* into another system *Q*.

This leads to the following statement of the principle of model adequacy:

Principle 1: Principle of Adequacy:

A domain adequately satisfies a formal theory if and only if all the properties of the system are satisfied by it (Mancuso et al., 2001, p. 84).

As a result, any domain that fails to interpret and legitimise the formal theory is insufficient for it to serve as a model. The problem of the intuitionistic model for number theory is rooted in the refusal of its proponents to give legitimacy to some of the properties of classical mathematics, like zero *(0)*. In its attempt to purify mathematics, intuitionism appears to be too drastic a model for the theory. The logicist problem with respect to adequacy is rooted in their refusal to acknowledge the essential position ordinality occupies in number theory, beyond simply satisfying arithmetic. Such a refusal is responsible for the complexity of their axioms. Some other authors refer to the principle referred to as adequacy in this study as truth functional completeness (Mancuso et al., 2001, p. 84).Truth functional completeness is however, better called a converse adequacy.

Principle 2: The converse principle of adequacy:

A formal theory is adequate for a domain if all the elements of such a domain are finitely or infinitely interpretable in it without contradiction (Mancuso et al., 2001, p. 84).

The completeness of a formal system is the formalisation of the consistency of the system within it (Gödel, 1986, p. 18). Gödel has shown that no uninterpreted formal system is complete, if the system is ω-consistent, then it is (simply) incomplete. ω-consistency in Gödel means formal consistency alone. (Gödel, 1995, p.9). The same theorem has been stated by Krajewski as follows:

"There exists a sentence *Gs* of the theory *S* such that *(*)* if *S* is consistent then *Gs* is not provable in *S*, even though *Gs* is true" (p.40).

Formal completeness is therefore a property of the availability of a model for the formal system. Hence, all countable models of Peano arithmetic are complete. Due to the fact that a countable model of any arithmetic assumes the intuitive proof of its entities, the consistency of the system, which does not need to be formalised in the system, is assumed intuitively and therefore proved by realisations (Gödel, 1990, p. 27).

With respect to propositional calculus, it has been noted by some writers, like Alfred Tarski, that its model is the truth table (Tarski, 1983, p. 13). Thus, an adequate and complete propositional calculus is determined by its capacity to express all possible truth tables. If the notion of truth is relative to that of being or "being the

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under Volume: 3 Issue: 12 in Dec-2022 https://grnjournals.us/index.php/AJSHR

case," then the propositional calculus is modelled by general ontology. After all, Aristotle had alleged that logic expresses the property of being (Evans 45).

Conclusion

What has been achieved so far is an exposition of the notions of axiomatic method and model theory. It is important to note that the study was able to present a demonstration of how the axiomatic method and its various applications have evolved over the course of history. Another important achievement is the proof that model theory is a necessary corollary of the axiomatic method, because this has gone to show that formalism is not just a game, as some authors have supposed. But all in all, the work has been able to take the lead in proposing an in-depth analysis of key concepts in mathematical logic before their application in system building and analysis.

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