AMERICAN JOURNAL OF SOCIAL AND HUMANITARIAN RESEARCH

ISSN: 2690-9626
Vol.3, No 3, 2022

Solvability of a Boundary Value Problem for a Fourth-Order Mixed Type Equation

Jamila Amanbaevna Otarova, Islam Bawenovich Saliev
Karakalpak state university named after Berdakh, Nukus, Uzbekistan

Annotation: In a rectangular domain, we study the boundary value problem for a fourth-order mixed differential equation of mixed type containing a wave operator and the product of the inverse and direct heat conduction operators. The criteria for the uniqueness of the solution of the problem, which are constructed as the sum of a Fourier series, are established. The stability of the obtained solution and the strong solvability of the problem are proved.

Keywords: equation of mixed type, Fourier series, completeness, regular solution, stability, strong solution.

Introduction. Various boundary value problems for individual types of differential equations in partial derivatives and for equations of mixed type of the fourth order have been studied in many papers. In [1], questions of the complete classification and reduction to canonical form of fourth-order linear partial differential equations were studied. Also, correct boundary-value problems for hyperbolic and mixed types equations were posed and investigated. Direct and inverse boundary value problems for fourth-order equations are studied in [2–8]. In the present work for the equation

$$Lu \equiv \begin{cases} u_{xx} - u_{tt} = f_1(x,t), \ \in \Omega^+, \\ u_{xxxx} - u_{tt} = f_2(x,t), \in \Omega^-, \end{cases} \quad (1)$$

where $f_1(x,t), f_2(x,t)$ – are the specified functions. In the $\Omega = \{(x,t): 0 < x < p, \ -T_1 < t < T_2\}$, area where $\Omega^+ = \Omega \cap (t > 0), \Omega^- = \Omega \cap (t < 0)$, the boundary problem is investigated.

Formulation. Problem A. Find a function $u(x,t)$, such that:

1) is continuous in $\overline{\Omega}$, together with its derivatives given in the boundary conditions;

2) is a regular solution of equation (1) in $\Omega^+ \cup \Omega^-$;

3) satisfies the boundary conditions:

$$u(0,t) = u(p,t) = 0, \quad -T_1 \le t \le T_2, \quad (2)$$

$$u_{xx}(0,t) = u_{xx}(p,t) = 0, \quad -T_1 \le t \le 0, \quad (3)$$

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 2 in February-2022 https://grnjournals.us/index.php/AJSHR

$$u(x,T_2)=0, \qquad 0\le x\le p, \qquad (4)$$

$$u(x,-T_1)=0, \qquad 0\le x\le p, \qquad (5)$$

4) satisfies the gluing condition

$$u_t(x,+0)=u_t(x,-0), \qquad 0<x<p. \qquad (6)$$

We introduce the notation:

$$W_1(\Omega^+)=\left\{f_1(x,t): f_1\in C^{3,0}_{x,t}\left(\overline{\Omega}^+\right),\ f_{1xxxx}\in L_2(\Omega^+), \forall t\in[0,T_2], f_1(0,t)=f_1(p,t)=0, f_{1xx}(0,t)=f_{1xx}(p,t)=0\right\},$$

$$W_2(\Omega^-)=\left\{f_2(x,t): f_2\in C^{2,0}_{x,t}\left(\overline{\Omega^-}\right),\ f_{2xxx}\in L_2(\Omega^-), \forall t\in[-T_1,0], f_2(0,t)=f_2(p,t)=0,\ f_{2xx}(0,t)=f_{2xx}(p,t)=0\right\}.$$

$$V(\Omega)=\left\{u(x,t): u\in C(\overline{\Omega}), u_{xx}, u_{tt}\in C(\Omega^+), u_{xx}\in C\left(\overline{\Omega^-}\right), u_{xxxx}, u_{tt}\in C(\Omega^-), u_t\in C(\Omega),\right.$$

conditions are met $(2)-(6)\}$.

$$f(x,t)=\begin{cases} f_2(x,t), & -T_1\le t\le 0,\\ f_1(x,t), & 0\le t\le T_2.\end{cases} \qquad W(\Omega)=\begin{cases} W_1(\Omega^+), & 0\le t\le T_2,\\ W_2(\Omega^-), & -T_1\le t\le 0.\end{cases}$$

Definition 1. A function $u(x,t)\in V(\Omega)$ is called a *regular solution* of problem A, for, $f(x,t)\in C(\Omega)$ if it satisfies equation (1) in Ω.

Definition 2. A function $u(x,t)\in L_2(\Omega)$ is called a *strong solution* of problem A for $f(x,t)\in L_2(\Omega)$ if there is a sequence $\{u_k\}$, $k=1,2,...$

regular solutions such that $\|u_k-u\|_{L_2(\Omega)}\to 0$, $\|Lu_k-f\|_{L_2(\Omega)}\to 0$ for $k\to\infty$.

We define operator $L: V(\Omega)\to C(\Omega)$ on the set $V(\Omega)$. By virtue of the relation $C_0^\infty(\Omega)\subset V(\Omega)\subset L_2(\Omega)$, the domain of definition $D(L)\equiv V(\Omega)$ of the operator L is dense in $L_2(\Omega)$. The regular solvability of problem A is equivalent to the solvability of the operator equation

$$Lu=f$$

where L works by the rule (1), $\forall u\in V(\Omega)$

Results and discussion.

Theorem 1. Let the numbers p and T_2 such that for $n=1,2...$

$$N_n(T)\equiv\left|\left(1-e^{-2\frac{n^2\pi^2}{p^2}T_1}\right)\cdot\cos\frac{n\pi}{p}T_2+\frac{n\pi}{p}\left(1+e^{-2\frac{n^2\pi^2}{p^2}T_1}\right)\cdot\sin\frac{n\pi}{p}T_2\right|\ne 0, \quad (7)$$

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 2 in February-2022 https://grnjournals.us/index.php/AJSHR

then if there is a regular solution to problem A, then it is unique.

Theorem 2. If $f_1(x,t) \in W_1(\Omega^+)$, $f_2(x,t) \in W_2(\Omega^-)$ and $N_n(T) \geq \delta_0 > 0$ for $n = 1, 2, \ldots,$ then a regular solution of problem A exists and is stable.

We regularly look for a solution to the problem in the form of a series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \cdot X_n(x), \qquad (8)$$

where $X_n(x)$ is determined by system

$$X_n(x) = \sqrt{\frac{2}{p}} \sin \lambda_n x, \quad \lambda_n = \frac{n\pi}{p}, \quad n = 1, 2, \ldots, \qquad (9).$$

Substituting (8) into equation (1) and decomposing the right-hand sides of this equation into a Fourier series in functions $X_n(x)$

$$f_1(x,t) = \sum_{n=1}^{\infty} f_{1n}(t) \cdot X_n(x), \quad \text{where } f_{1n}(t) = \int_0^p f_1(x,t) \cdot X_n(x),$$

$$f_2(x,t) = \sum_{n=1}^{\infty} f_{2n}(t) \cdot X_n(x), \quad \text{where } f_{2n}(t) = \int_0^p f_2(x,t) \cdot X_n(x),$$

come to the following two equations

$$u_n''(t) + \lambda_n^2 u_n''(t) = -f_{1n}(t), \; t > 0,$$

$$u_n''(t) - \lambda_n^4 u_n(t) = -f_{2n}(t), \; t < 0.$$

By solving these equations by the method of variation of arbitrary constants and applying the condition 1) of $u_n(+0) = u_n(-0)$, of the problem and (4) - (6), we get

$$u_n(t) = \frac{\sin \lambda_n (T_2 - t)}{\lambda_n \left(\cos \lambda_n T_2 \cdot sh\lambda_n^2 T_1 + \lambda_n \sin \lambda_n T_2 \cdot ch\lambda_n^2 T_1\right)} \int_{-T_1}^{0} sh\lambda_n^2 (T_1 + \tau) \cdot f_{2n}(\tau) d\tau +$$

$$+ \int_0^{T_2} K_{n1}(t,\tau) f_{1n}(\tau) d\tau, \quad t > 0, \qquad (10)$$

$$u_n(t) = \frac{sh\lambda_n^2 (T_1 + t)}{\lambda_n \left(\cos \lambda_n T_2 \cdot sh\lambda_n^2 T_1 + \lambda_n \sin \lambda_n T_2 \cdot ch\lambda_n^2 T\right)_1} \int_0^{T_2} \sin \lambda_n (T_2 - \tau) \cdot f_{1n}(\tau) d\tau +$$

$$+ \frac{1}{\lambda_n} \int_{-T_1}^{0} K_{n2}(t,\tau) f_{2n}(\tau) d\tau, \qquad t < 0, \qquad (11)$$

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 2 in February-2022 https://grnjournals.us/index.php/AJSHR

where

$$K_{n1}(t,\tau)=\begin{cases}\dfrac{\cos\lambda_n\tau\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_n\tau\cdot ch\lambda_n^2T_1}{\lambda_n\left(\cos\lambda_nT_2\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2T_1\right)}\cdot\sin\lambda_n\left(T_2-t\right),\ 0\le\tau\le t,\\ \dfrac{\cos\lambda_nt\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nt\cdot ch\lambda_n^2T_1}{\lambda_n\left(\cos\lambda_nT_2\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2T_1\right)}\cdot\sin\lambda_n\left(T_2-\tau\right),\ t\le\tau\le T_2,\end{cases}\quad t>0,\quad (12)$$

$$K_{n2}(t,\tau)=\begin{cases}\dfrac{\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2t-\cos\lambda_nT_2\cdot sh\lambda_n^2t}{\lambda_n\cdot\left(\cos\lambda_nT_2\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2T_1\right)}\cdot sh\lambda_n^2\left(T_1+\tau\right),\ -T_1\le\tau\le t,\\ \dfrac{\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2\tau-\cos\lambda_nT_2\cdot sh\lambda_n^2\tau}{\lambda_n\cdot\left(\cos\lambda_nT_2\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2T_1\right)}\cdot sh\lambda_n^2\left(T_1+t\right),\ t\le\tau\le 0.\end{cases}\quad t<0.\quad (13)$$

We introduce the notation

$$F_{n1}(t,\tau)=\begin{cases}\dfrac{\sin\lambda_n\left(T_2-t\right)\cdot sh\lambda_n^2\left(T_1+\tau\right)}{\cos\lambda_nT_2\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2T_1}, & -T_1\le\tau\le 0,\\ K_{n1}(t,\tau), & 0\le\tau\le T_2,\end{cases}\quad 0\le t\le T_2,\ (14)$$

$$F_{n2}(t,\tau)=\begin{cases}K_{n2}(t,\tau), & -T_1\le\tau\le 0,\\ \dfrac{\sin\lambda_n\left(T_2-\tau\right)\cdot sh\lambda_n^2\left(T_1+t\right)}{\cos\lambda_nT_2\cdot sh\lambda_n^2T_1+\lambda_n\sin\lambda_nT_2\cdot ch\lambda_n^2T_1}, & 0\le\tau\le T_2,\end{cases}\quad -T_1\le t\le 0,\ (15)$$

$$f_n(\tau)=\begin{cases}f_{2n}(\tau), & -T_1\le\tau\le 0,\\ f_{1n}(\tau), & 0\le\tau\le T_2.\end{cases}$$

By virtue of the introduced notation, (10), (11) can be represented as follows

$$u_n(t)=\frac{1}{\lambda_n}\int_{-T_1}^{T_2}F_{n1}(t,\tau)\,f_n(\tau)\,d\tau,\quad t>0,\qquad (16)$$

$$u_n(t)=\frac{1}{\lambda_n}\int_{-T_1}^{T_2}F_{n2}(t,\tau)\,f_n(\tau)\,d\tau,\quad t<0.\qquad (17)$$

This is a formal solution for the problem A in the field Ω.

Conclusion.

Lemma 1. If $N_n(T)\ge\delta_0>0,$ if $n=1,2...$ then the estimates are true

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 2 in February-2022 https://grnjournals.us/index.php/AJSHR

$$\left|u_n(t)\right| \le \frac{C_2}{\delta_0}\left[\left\|f_{1n}\right\|_{L_2(0;T_2)} + \frac{1}{\lambda_n}\left\|f_{2n}\right\|_{L_2(-T_1;0)}\right], \quad t>0, \tag{18}$$

$$\left|u_n'(t)\right| \le \frac{C}{\delta_0}\left[\lambda_n\left\|f_{1n}\right\|_{L_2(0;T_2)} + \left\|f_{2n}\right\|_{L_2(-T;0)}\right], \qquad t>0,$$

$$\left|u_n''(t)\right| \le \frac{C}{\delta_0}\left[\lambda_n^2\left\|f_{1n}\right\|_{L_2(0,T_2)} + \lambda_n\left\|f_{2n}\right\|_{L_2(-T_1;0)}\right] + C_1\left\|f_1\right\|_{C(\bar{\Omega})}, \quad t>0$$

and

$$\left|u_n(t)\right| \le \frac{C_2}{\lambda_n}\left[\left\|f_{1n}\right\|_{L_2(0;T_2)} + \frac{1}{\lambda_n}\left\|f_{2n}\right\|_{L_2(-T_1;0)}\right], \quad t<0, \tag{19}$$

$$\left|u_n'(t)\right| \le \frac{C}{\delta_0}\left[\lambda_n\left\|f_{1n}\right\|_{L_2(0;T_2)} + \left\|f_{2n}\right\|_{L_2(-T;0)}\right]; \qquad t<0,$$

$$\left|u_n''(t)\right| \le \frac{C\lambda_n^3}{\delta_0}\left[\left\|f_{1n}\right\|_{L_2(0,T_2)} + \left\|f_{2n}\right\|_{L_2(-T_1;0)}\right] + C\left\|f_2\right\|_{C(\bar{\Omega})}, \quad t<0$$

where $C, C_1, C_2 = const > 0.$

Lemma 2. A regular solution of problem A satisfies the estimate

$$\|u\|_{L_2(\Omega)} \le C_3\|f\|_{L_2(\Omega)}, \tag{20}$$

where $C_3 > 0 -$ is a constant number depending only on the size of the domain Ω and independent of the function $u(x,t)$.

Theorem 3. For any $f \in L_2(\Omega)$ strong solution of problem A , there exists uniquely, stably, satisfies the estimate (20) and is given by formula

$$u(x,t) = \begin{cases} \sum\limits_{n=1}^{\infty} \dfrac{X_n(x)}{\lambda_n} \int\limits_{-T_1}^{T_2} F_{n1}(t,\tau) f_n(\tau)\, d\tau, & t>0, \\ \sum\limits_{n=1}^{\infty} \dfrac{X_n(x)}{\lambda_n} \int\limits_{-T_1}^{T_2} F_{n2}(t,\tau) f_n(\tau)\, d\tau, & t<0 \end{cases}$$

References

1. Dzhuraev, T.D., Sopuev, A. On the theory of fourth-order partial differential equations. - Tashkent: Fan. 2000. - 144 p.
2. Amanov D., Otarova J.A. A boundary value problem for a fourth-order mixed-type equation // Uzb. mat. journals - Tashkent, 2008. - №3. - C.13-22.

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 2 in February-2022 https://grnjournals.us/index.php/AJSHR

3. Kadirkulov B.Zh. On an inverse problem for a fourth-order parabolic equation // Uzbek Mathematical Journal. - Tashkent, 2012. –№1. - C. 74-80.
4. Kadirkulov B., Zh. On a nonlocal problem for a fourth-order parabolic equation with a fractional Dzhrbashyan-Nersesyan operator. // Differential Equations - Minsk, 2016. T.- 52. No. 1.- C.123–127.
5. J.A. Otarova Solvability and spectral properties of a self-adjoint problem for a fourth-order equation // Dokl. AN RUZ. Tashkent, 2008 - № 1.- p. 10-14.
6. Pyatkov S.G. Solvability of a boundary value problem for fourth-order pseudo parabolic equations // Vestnik NGU. Series. Maths. 2005. T.5. issue 3. P. 43-56.
7. Rustamova L.R. Nonlocal boundary-value problems of Bitsadze-Samarsky type for a fourth-order mixed-type equation // Izvestiya Vuzov. Sev.- Kav. reg. Estes.nauki. - 2007. - №4. - P. 14-16.
8. Saadalov T.Y. Boundary value problems for a pseudo parabolic-hyperbolic equation of the fourth order // Natural and mathematical sciences in the modern world: Coll. Art. on mater. XIII Intern. Scientific Pract. Conf. № 5 (40) .- Novosibirsk: Sibak, 2016.- p. 138-145.
9. Moiseev E.I. On the solution of a single nonlocal boundary value problem by the spectral method // Differential Equations. - Kiev, 1999.- № 8 (35) - p. 1094-1100.

ISSN 2690-9626 (online), Published by "Global Research Network LLC"
under Volume: 3 Issue: 2 in February-2022 https://grnjournals.us/index.php/AJSHR